SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2008

YEAR 12

ASSESSMENT TASK #2

Mathematics Extension 1

General Instructions

- Working time – 90 Minutes
- Reading Time – 5 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each **section** is to be returned in a separate bundle.
- All necessary working should be shown in every question if full marks are to be awarded.
- Full marks may not be awarded for untidy or badly arranged work.

Total Marks – 80

- Attempt questions 1 – 3
- All questions are NOT of equal value.

Examiner: *R.Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Section A

(Start a new booklet.)

Question 1. (28 marks)

Marks

(a) Solve for x, leaving your answer in exact form: **3**

$$\ln x = \frac{1}{\ln x}$$

(b) Find the first derivative of $x^2 e^{2x}$. **3**

(c) Find the value of k if **2**

$$\int_1^k \sqrt{x}\, dx = \frac{14}{3}$$

(d) Solve for x, leaving your answer in exact form: **3**

$$\log_{\sqrt{a}}(x+2) - \log_{\sqrt{a}}(2) = \log_{\sqrt{a}}(x) + \log_{\sqrt{a}}(2)$$

(e) Differentiate the following with respect to x: **4**

(i) $\sin^{-1}(3x+2)$

(ii) $\dfrac{\tan^{-1} x}{1+x^2}$

(f) Find an indefinite integral of each of the following (with respect to x): **4**

(i) $\dfrac{1}{\sqrt{4-x^2}}$

(ii) $\dfrac{1}{9+4x^2}$

(g) Using the fact that $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$, and without using a calculator, show that $\tan^{-1}\left(\dfrac{1}{4}\right) + \tan^{-1}\left(\dfrac{3}{5}\right) = \dfrac{\pi}{4}$. **4**

(h) Find $6\pi \int \cos(2\pi x - 1)\, dx$. **3**

Section continued overleaf.

Question 1 (cont.)

Marks

(i) The letters of the word *CALCULUS* are arranged in a row. How many different arrangements are possible? **2**

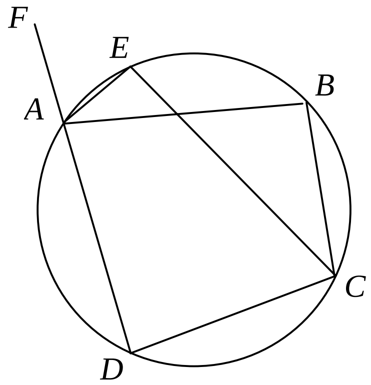
End of Section A

Section B
(Start a new booklet.)

Question 2. (26 marks)

Marks

(a) In the diagram at right, DA is produced to F, and EC bisects $\angle BCD$. **4**



(i) Copy the diagram to your answer booklet.

(ii) Prove that AE bisects $\angle FAB$.

(b) Consider the function $y = 4\cos^{-1}\left(\frac{x}{3}\right)$. **6**

(i) Find the domain and range of the function $y = 4\cos^{-1}\left(\frac{x}{3}\right)$.

(ii) Sketch the graph of the function $y = 4\cos^{-1}\left(\frac{x}{3}\right)$ showing clearly the intercepts on the coordinate axes, and the coordinates of any endpoints.

(iii) Find the area of the region in the first quadrant bounded by the curve $y = 4\cos^{-1}\left(\frac{x}{3}\right)$ and the coordinate axes.

(c) The area between the curve $y = \ln x$, the x-axis, and the lines $x = 2$ and $x = 4$ is rotated about the x-axis. Use Simpson's Rule with three function values to estimate the volume of the solid so formed. Give your answer correct to two decimal places. **4**

(d) Seven chairs (two of which are identical) are arranged in a circle. How many different arrangements are possible? **2**

(e) Evaluate $\int_0^{\frac{\pi}{4}} \sin^2\theta \, d\theta$ leaving your answer in exact form. **3**

(f) The equation $\sin x = 1 - 2x$ has a root near $x = 0{\cdot}3$. **4**

(i) Use one application of Newton's Method to obtain another approximation to the root.

(ii) Which of the two approximations to the root is better, and why?

Section Continued Overleaf.

(g) (i) Sketch the graph of $y = 1 - 3\cos 2x$ in the domain $-\pi \le x \le \pi$. **2**

(ii) How many solutions to the equation $1 - 3\cos 2x = 5$ exist in the domain $-\pi \le x \le \pi$? Justify your answer. **1**

End of Section B

Section C
(Start a new booklet.)

Question 3 (26 marks)

Marks

(a) Evaluate $\lim_{x \to 0} \frac{x}{\sin 5x}$. **1**

(b) (i) Show that $\frac{5}{(x-2)(x+3)}$ can be expressed in the form $\frac{1}{x-2} - \frac{1}{x+3}$. **4**

(ii) Hence or otherwise find $\int \frac{5dx}{(x-2)(x+3)}$.

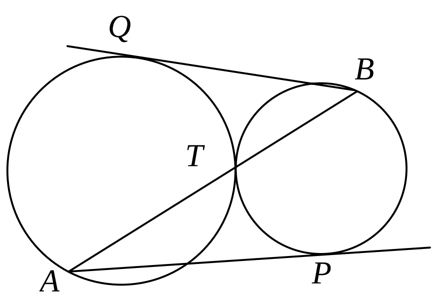
(c) A motorway pay station has five toll gates, three of which are automatic, and two of which are manually operated. Drivers with exact money may use any one of the five gates, but drivers requiring change must use a manually operated gate. **4**

A Suzuki driver, an Alfa driver, and a Holden driver use the motorway every day.

(i) On one day the Suzuki driver requires change, and the other two have exact money. Find the number of ways in which the three drivers can go through the pay station so that each uses a different gate.

(ii) On another day all three drivers have the exact money. Find the number of ways they can go through the pay station so that exactly one uses a manual gate, and each uses a separate gate.

(d) In the diagram at right, the circles touch at T, and ATB is a straight line. **3**



AP is a tangent to the circle PTB, while BQ is a tangent to the circle QTA.

(i) Copy the diagram to your answer sheet.

(ii) Prove that $(AP)^2 + (BQ)^2 = (AB)^2$

Section Continued Overleaf.

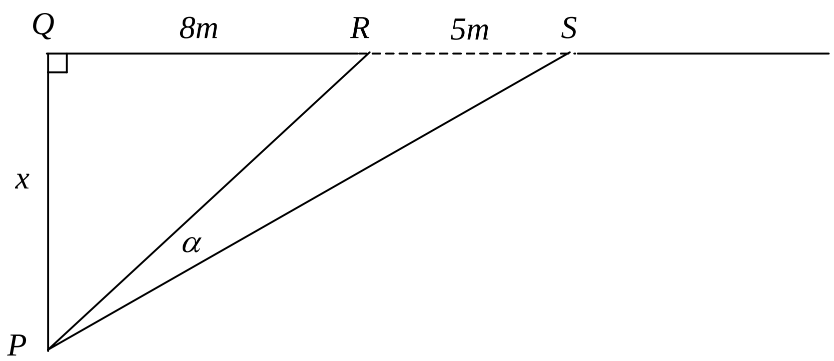
(e) Consider the function $f(x) = e^x - 4$. **6**

(i) On a large diagram sketch the graph of $f(x)$ clearly showing the coordinates of any intersections with the axes, and state the equations of any asymptotes.

(ii) On the same diagram as above, sketch the graph of the inverse function $f^{-1}(x)$ clearly showing the coordinates of any intersections with the axes, and state the equations of any asymptotes.

(iii) Explain why the *x*-coordinate of any point of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 4 = 0$.

(f) **8**



Ron *The Demolisher* is attacking a fortress with arrows from his position *P* behind the wall *QP* running out at rightangles to the fortress wall *QRS*. Ron is x metres from the fortress and has an angle of vision of α through opening *RS*.

(i) Using the measurements on the diagram, show that the angle of vision is given by $\alpha = \tan^{-1}\left(\frac{13}{x}\right) - \tan^{-1}\left(\frac{8}{x}\right)$.

(ii) Find the exact value of *x* in order to give the maximum angle of vision.

(iii) Hence find the maximum angle of vision, in radians (correct to two decimal places).

End of Section C

This is the end of the paper.

MATHEMATICS EXTENSION 1

YEAR 12 ASS TASK 2 2008

Question 1:

(a) $\ln x = \frac{1}{\ln x}$

$(\ln x)^2 = 1$

$\ln x = \pm 1$

$x = e$ or $\frac{1}{e}$ (3)

(b) $\frac{d}{dx}(x^2 e^{2x}) = e^{2x}.2x + x^2.2e^{2x}$

$= 2xe^{2x}(1 + x)$ (3)

(c) $\int_1^K \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^K = \frac{14}{3}$

$\therefore \frac{2}{3}(K^{3/2} - 1) = \frac{14}{3}$

$\therefore K^{3/2} - 1 = 7$

$\therefore K^{3/2} = 8$

$\therefore K^{1/2} = 2$

$\therefore K = 4$ (2)

(d) $\log_{\sqrt{a}}(x+2) - \log_{\sqrt{a}}(2) = \log_{\sqrt{a}} x + \log_{\sqrt{a}} 2$

$\therefore \log_{\sqrt{a}} \frac{x+2}{2} = \log_{\sqrt{a}} 2x$

$\therefore \frac{x+2}{2} = 2x$

$\therefore x + 2 = 4x$

$\therefore 3x = 2$

$\therefore x = \frac{2}{3}$ (3)

(e) (i) $\frac{d}{dx}\left(\sin^{-1}(3x+2)\right) = \frac{1}{\sqrt{1-(3x+2)^2}}.3$

$= \frac{3}{\sqrt{1 - 9x^2 - 12x - 4}}$

$= \frac{3}{\sqrt{-9x^2 - 12x - 3}}$ (2)

(ii) $\frac{d}{dx}\left(\frac{\tan^{-1}x}{1+x^2}\right) = \frac{(1+x^2).\frac{1}{1+x^2} - \tan^{-1}x.2x}{(1+x^2)^2}$

$= \frac{1 - 2x\tan^{-1}x}{(1+x^2)^2}$ (2)

(f) (i) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + c$ (1)

(ii) $\int \frac{1}{9+4x^2}\,dx = \frac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx$

$= \frac{1}{4} \times \frac{2}{3}\tan^{-1}\frac{2x}{3} + c$

$= \frac{1}{6}\tan^{-1}\frac{2x}{3} + c$ (3)

(g) $\tan\left(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5}\right)$

$= \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4} \times \frac{3}{5}}$

$= \frac{\frac{17}{20}}{1 - \frac{3}{20}}$

$= 1$

$\therefore \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5} = \frac{\pi}{4}$ (4)

NOTE: $\tan^{-1}\frac{1}{4}$ and $\tan^{-1}\frac{3}{5}$ are both acute angles less than $\frac{\pi}{4}$ and thus $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5} < \frac{\pi}{2}$

(h) $6\pi \int \cos(2\pi x - 1)\,dx$

$= 6\pi . \frac{1}{2\pi}\sin(2\pi x - 1) + c.$ (3)

$= 3\sin(2\pi x - 1) + c.$

(i) No of arrangements $= \frac{8!}{2!\,2!\,2!}$

$= 7!$

$= 5040$ (2)

28

QUESTION 2

a ii) Let $\angle EAB = x^\circ$

$\angle BCE = x^\circ$ (angles in the same segment).

$\angle ECD = x^\circ$ (EC bisects $\angle BCD$).

$\angle FAE = x^\circ$ (exterior angle of a cyclic quad ABCD).

$\therefore \angle EAB = \angle FAE$

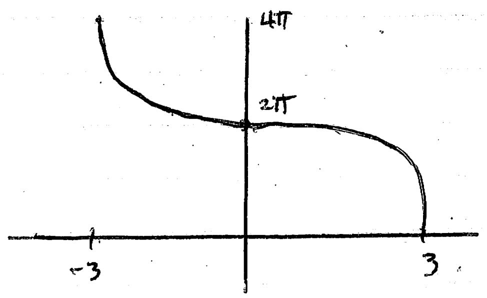
$\therefore$ AE bisects $\angle FAB$.

b i) Domain: $-1 \leq \frac{x}{3} \leq 1$

$-3 \leq x \leq 3$

Range: $0 \leq \frac{y}{4} \leq \pi$

$0 \leq y \leq 4\pi$.

ii) 

iii) $\int_0^3 4\cos^{-1}\frac{x}{3}\,dx = \int_0^{2\pi} 3\cos\frac{y}{4}\,dy$

$= \left[12\sin\frac{y}{4}\right]_0^{2\pi}$

$= 12\sin\frac{\pi}{2} - 12\sin 0$

$= 12$ units^2

c) $\pi \int_2^4 (\ln x)^2 \, dx = \pi \frac{4-2}{6}\left[(\ln 2)^2 + 4\left(\ln \frac{2+4}{2}\right)^2 + (\ln 4)^2\right]$

$= \frac{\pi}{3}\left((\ln 2)^2 + 4(\ln 3)^2 + (\ln 4)^2\right)$

$= 7.57 \text{ units}^3$

d) $\frac{6!}{2} = 360$

e) $\int_0^{\frac{\pi}{4}} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - \cos 2\theta \, d\theta$

$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{\sin \frac{\pi}{2}}{2}\right)$

$= \frac{\pi}{8} - \frac{1}{4}$

f i) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(x) = \sin x - 1 + 2x$

$f'(x) = \cos x + 2$

$x_1 = 0.3 - \frac{f(0.3)}{f'(0.3)}$

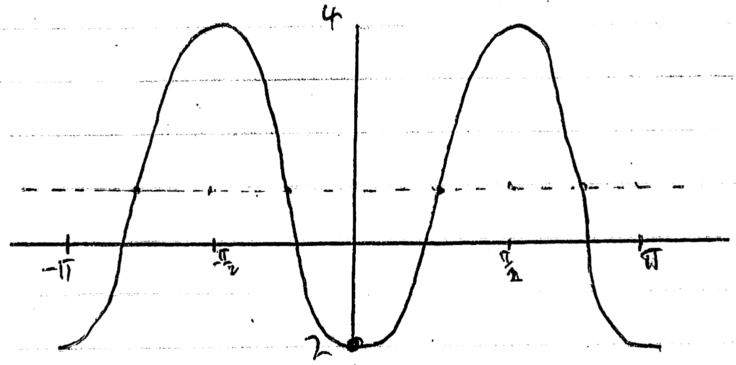
$= 0.335$

ii) $f(0.3) = -0.1045$

$f(0.335) = 0.001231$

Second approx. is the better.

g i)



ii) None. Range: $-2 \leq y \leq 4$.

$y=5$ is outside the range.

2008 Assessment #2 Mathematics Extension 1:
Solutions— Section C

3. (a) Evaluate $\lim_{x\to 0} \dfrac{x}{\sin 5x}$. [1]

Solution:
$$\begin{aligned}\lim_{x\to 0} \frac{x}{\sin 5x} &= \lim_{x\to 0} \frac{5x}{\sin 5x} \times \frac{1}{5}, \\ &= 1 \times \frac{1}{5}, \\ &= \frac{1}{5}.\end{aligned}$$

(b) (i) Show that $\dfrac{5}{(x-2)(x+3)}$ can be expressed in the form $\dfrac{1}{x-2} - \dfrac{1}{x+3}$. [4]

Solution:
$$\begin{aligned}\frac{1}{x-2} - \frac{1}{x+3} &= \frac{(x+3)-(x-2)}{(x-2)(x+3)}, \\ &= \frac{5}{(x-2)(x+3)}.\end{aligned}$$

(ii) Hence or otherwise find $\displaystyle\int \frac{5dx}{(x-2)(x+3)}$.

Solution:
$$\begin{aligned}\int \frac{5dx}{(x-2)(x+3)} &= \int \frac{dx}{x-2} - \int \frac{dx}{x+3}, \\ &= \ln(x-2) - \ln(x+3) + c, \\ &= \ln\left(\frac{x-2}{x+3}\right) + c.\end{aligned}$$

(c) A motorway pay station has five toll gates, three of which are automatic, and two of which are manually operated. Drivers with exact money may use any one of the five gates, but drivers requiring change must use a manually operated gate. [4]
A Suzuki driver, an Alfa driver, and a Holden driver use the motorway every day.

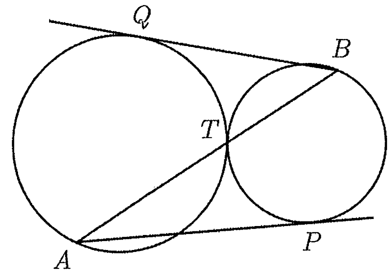
(i) On one day the Suzuki driver requires change, and the other two have exact change. Find the number of ways in which the three drivers can go through the pay station so that each uses a different gate.

Solution: Suzuki can choose 2 gates. This leaves any of 4 gates for the next car and then any of 3 gates for the last car: $2 \times 4 \times 3 = 24$

(ii) On another day all three drivers have the exact money. Find the number of ways they can go through the pay station so that exactly one uses a manual gate, and each uses a separate gate.

Solution: 3 ways to choose who goes manual and 2 manual gates implies 6 ways. This leaves any of 3 automatic gates for the next car and then either of 2 gates for the last car: $3 \times 2 \times 3 \times 2 = 36$

(d) In the diagram at right, the circles touch at T, and ATB is a straight line. AP is a tangent to the circle PTB, while BQ is a tangent to the circle QTA. **3**



(i) Copy the diagram to your answer sheet.

(ii) Prove that $(AP)^2 + (BQ)^2 = (AB)^2$.

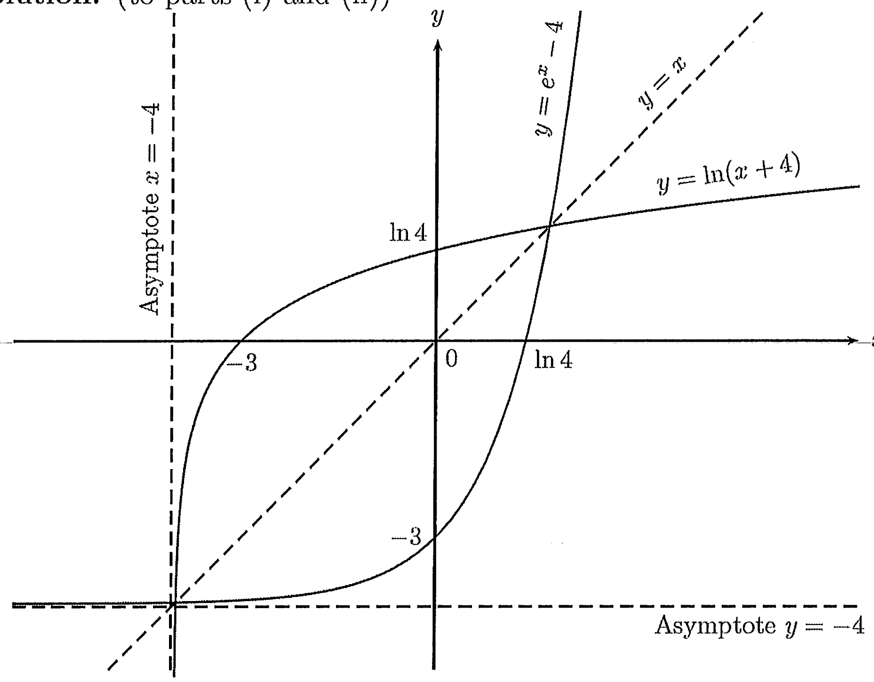
Solution:
$$\begin{aligned} BQ^2 &= BT.BA \text{ (intersecting tangent-secant theorem)},\\ AP^2 &= AT.AB \text{ (intersecting tangent-secant theorem)},\\ AP^2 + BQ^2 &= AB.AT + AB.TB,\\ &= AB(AT + TB),\\ &= AB.AB,\\ &= AB^2. \end{aligned}$$

(e) Consider the function $f(x) = e^x - 4$. **6**

(i) On a large diagram sketch the graph of $f(x)$ clearly showing the coordinates of any intersections with the axes, and state the equations of any asymptotes.

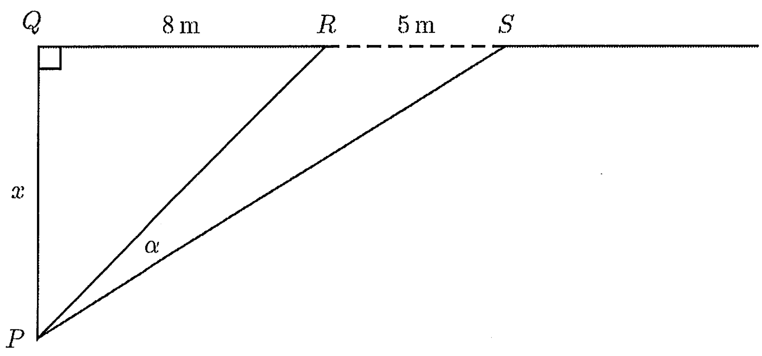
(ii) On the same diagram as above, sketch the graph of the inverse function $f^{-1}(x)$ clearly showing the coordinates of any intersections with the axes, and state the equations of any asymptotes.

Solution: (to parts (i) and (ii))



(iii) Explain why the x-coordinate of any point of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 4 = 0$.

Solution: Both $f(x)$ and $f^{-1}(x)$ intersect on the line $y = x$.
$\therefore\ e^x - 4 = x$ at the intersections,
i.e. $e^x - x - 4 = 0$ is satisfied by the intersection points.

(f) 

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Ron *The Demolisher* is attacking a fortress with arrows from his position P behind the wall QP running out at right-angles to the fortress wall QRS. Ron is x metres from the fortress and has an angle of vision of α through opening RS.

(i) Using the measurements on the diagram, show that the angle of vision is given by $\alpha = \tan^{-1}\left(\frac{13}{x}\right) - \tan^{-1}\left(\frac{8}{x}\right)$.

Solution:
$$\begin{aligned} Q\widehat{P}R + R\widehat{P}S &= Q\widehat{P}S, \\ Q\widehat{P}R + \alpha &= Q\widehat{P}S, \\ \alpha &= Q\widehat{P}S - Q\widehat{P}R, \\ &= \tan^{-1}\left(\frac{13}{x}\right) - \tan^{-1}\left(\frac{8}{x}\right). \end{aligned}$$

(ii) Find the exact value of x in order to give the maximum angle of vision.

Solution: Put $\alpha = f(x) - g(x)$ where $f(x) = \tan^{-1} u$, $u = \frac{13}{x}$ and $g(x) = \tan^{-1} v$, $v = \frac{8}{x}$.

$$\begin{aligned} f'(x) &= \frac{1}{1+u^2} \times \frac{-13}{x^2}, \\ &= \frac{-13}{x^2+169}. \quad \text{Similarly } g'(x) = \frac{-8}{x^2+64}. \\ \therefore \frac{d\alpha}{dx} &= \frac{8}{x^2+64} - \frac{13}{x^2+169}, \\ &= \frac{8x^2+1352-13x^2-832}{(x^2+169)(x^2+169)}, \\ &= \frac{520-5x^2}{(x^2+169)(x^2+169)}, \\ &= 0 \text{ when } x^2 = 104. \end{aligned}$$

x	10	$\sqrt{104}$	11
$\frac{d\alpha}{dx}$	4.5×10^{-4}	0	-1.6×10^{-3}

$\therefore$ Maximum α at $x = 2\sqrt{26}$.

(iii) Hence find the maximum angle of vision in radians (correct to two decimal places).

Solution:
$$\begin{aligned} \alpha_{\max} &= \tan^{-1}\left(\frac{13}{2\sqrt{26}}\right) - \tan^{-1}\left(\frac{8}{2\sqrt{26}}\right), \\ &\approx 0.24. \end{aligned}$$